

MORSE THEORY ON GRAPHS

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ABSTRACT. Let Γ be a finite d -valent graph and G an n -dimensional torus. An “action” of G on Γ is defined by a map which assigns to each oriented edge, e , of Γ , a one-dimensional representation of G (or, alternatively, a weight, α_e , in the weight lattice of G . For the assignment, $e \rightarrow \alpha_e$, to be a schematic description of a “ G -action”, these weights have to satisfy certain compatibility conditions: the GKM axioms). As in [GKM] we attach to (Γ, α) an equivariant cohomology ring, $H_G(\Gamma) = H(\Gamma, \alpha)$. By definition this ring contains the equivariant cohomology ring of a point, $\mathbb{S}(\mathfrak{g}^*) = H_G(pt)$, as a subring, and in this paper we will use graphical versions of standard Morse theoretical techniques to analyze the structure of $H_G(\Gamma)$ as an $\mathbb{S}(\mathfrak{g}^*)$ -module.

1. INTRODUCTION

As Richard Stanley so astutely observes in [St], “The number of systems of terminology presently used in graph theory is equal, to a close approximation, to the number of graph theorists.” Our terminological conventions in this paper will be the following: Given a finite d -valent graph, Γ , we will denote by V_Γ the vertices of Γ and by E_Γ the oriented edges of Γ . For each oriented edge, e , we will denote by \bar{e} the edge obtained by reversing the orientation of e and by $i(e)$ and $t(e)$ the initial and terminal vertices of e . (Thus “ d -valent” means that for every vertex, p , there are exactly d edges with $i(e) = p$.)

Let G be a torus of dimension $n > 1$, and let ϱ be a function which associates to every oriented edge, e , of Γ , a one-dimensional representation, ϱ_e , of G , with character $\chi_e : G \rightarrow S^1$, and τ a function which associates to every vertex, p , of Γ , a d -dimensional representation, τ_p , of G . We will say that ϱ and τ define an *action* of G on Γ if they satisfy the three axioms below:

Axiom A1:

$$\tau_p \simeq \bigoplus_{i(e)=p} \varrho_e.$$

Axiom A2:

$$\varrho_{\bar{e}} \simeq \varrho_e^*.$$

Axiom A3:

Let G_e be the kernel of $\chi_e : G \rightarrow S^1$ and let $p = i(e)$ and $q = t(e)$. Then

$$\tau_p|_{G_e} \simeq \tau_q|_{G_e}.$$

The data (Γ, ρ, τ) can be viewed as the schematic description of a genuine G -action, namely an action of G on a d -dimensional complex manifold, M , having the following properties:

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- P1 : The fixed point set of G is V_Γ , and, for every $p \in V_\Gamma$, τ_p is the isotropy representation of G on the tangent space to M at p .
- P2 : Each edge, e , of G , corresponds to a G -invariant imbedded projective line, $\gamma : \mathbb{CP}^1 \rightarrow M_e \subset M$. The points $p = i(e) = \gamma([1 : 0])$ and $q = t(e) = \gamma([0 : 1])$ are the two fixed points for the action of G on M_e , and ϱ_e is the isotropy representation of G on the tangent space to M_e at p .

These properties are exhibited by a class of manifolds, called *GKM manifolds*, which were first introduced in [GKM] by Goresky, Kottwitz and MacPherson and have subsequently been studied in a number of recent papers ([GZ1], [GZ2], [KR], [LLY], [TW]) by ourselves and by others. By definition, a complex G -manifold, M , is *GKM* if, for every fixed point, p , of G , the weights, α_i , $i = 1, \dots, d$, of the isotropy representation of G on the tangent space to M at p are pairwise linearly independent: *i.e.* α_i and α_j are linearly independent if $i \neq j$. For such a manifold we define its *one-skeleton* to be the set of points, $p \in M$, for which the stabilizer group of p is of dimension at least $n - 1$. What Goresky, Kottwitz and MacPherson observe is that this one-skeleton consists of closed submanifolds of M on which G acts in a fixed point free fashion and imbedded \mathbb{CP}^1 's satisfying P1-P2.

In [GZ2] we proved a converse to this result. Let (Γ, ϱ, τ) be a graph-theoretical G -space in the sense described above. For $e \in E_\Gamma$ let α_e be the weight of the representation, ϱ_e , and assume that for every vertex, p , of Γ , the weights α_e , $i(e) = p$, of the representation, τ_p , are pairwise linearly independent. Then there exists a complex G -manifold, M_Γ , of GKM type, whose one-skeleton has the properties P1-P2. (In fact, M_Γ is canonically constructed from the data (Γ, ρ, τ) and is basically just a tubular neighborhood of this one-skeleton.)

From Axiom A1, it is clear that the function, ρ , determines the function, τ ; and ρ , in turn, is determined by the function

$$\alpha : E_\Gamma \rightarrow \mathfrak{g}^*, \quad e \rightarrow \alpha_e.$$

In the geometric realization of Γ which we described above this function tells us how each of the two-spheres in the one-skeleton is rotated about its axis of symmetry; so we will call this function the *axial function* of the G -action on Γ . The axioms, A1-A3, can easily be translated into axioms on this axial function (as we will do in Section 2).

By realizing (Γ, ρ, τ) as a G -manifold, M_Γ , we can attach to it a cohomology ring: the equivariant cohomology ring, $H_G(M_\Gamma)$. The constant map, $M_\Gamma \rightarrow pt$, makes this into a module over the ring, $H_G(pt) = \mathbb{S}(\mathfrak{g}^*)$; and we will denote by $H_G(\Gamma)$ the torsion-free part of this module. The questions we will be concerned with in this paper are:

1. Is this module a free $\mathbb{S}(\mathfrak{g}^*)$ -module ?
2. If so, how many generators does it have in each dimension ?
3. Is there a nice combinatorial description of these generators ?

We will now formulate these questions more precisely. We will henceforth abandon any pretense of being concerned with G -actions on manifolds and deal only with G -actions on graphs. Fortunately, there is a very beautiful description in [GKM] of $H_G(\Gamma)$ in terms of Γ and α which allows us to do this.

Fix a vector, $\xi \in \mathfrak{g}$, with the property that

$$\alpha_e(\xi) \neq 0 \tag{1.1}$$

for all $e \in E_\Gamma$. We will call such a vector *polarizing*, and denote by \mathcal{P} the set of polarizing vectors. By axiom A2

$$\alpha_{\bar{e}}(\xi) = -\alpha_e(\xi), \quad (1.2)$$

so we can orient Γ by assigning to each unoriented edge, the orientation which makes $\alpha_e(\xi)$ positive. We will denote this orientation by o_ξ . To be able to do Morse theory on Γ we make the following essential assumption.

Acyclicity axiom. For some ξ satisfying (1.1), the orientation, o_ξ , of Γ that we have just described has no oriented cycles.

We will say that (Γ, α) is ξ -*acyclic* if this axiom is satisfied.

Let $V = V_\Gamma$. The acyclicity assumption implies that one can find a function $\phi : V \rightarrow \mathbb{R}$ with the property that, for every oriented edge, e , of Γ , with initial vertex p and terminal vertex q , $\phi(p) < \phi(q)$. We will say that such a function is ξ -*compatible* and call any function with this property a *Morse function*. A canonical choice of such a ϕ is the following: For every vertex, p , consider all oriented paths in (Γ, o_ξ) with terminal point p and let $\phi(p)$ be the length of the longest of these paths. It is clear that for this function to be unambiguously defined, Γ has to satisfy the acyclicity assumption. By a small perturbation if necessary, we may assume, without loss of generality, that ϕ is injective.

Following [GKM] we define the *cohomology ring*, $H_G(\Gamma)$, of (Γ, α) , to be the subring of the graded ring

$$\text{Maps}(V, \mathbb{S}(\mathfrak{g}^*)) = \bigoplus_{k \geq 0} \text{Maps}(V, \mathbb{S}^k(\mathfrak{g}^*)) \quad (1.3)$$

consisting of all maps, $f : V \rightarrow \mathbb{S}(\mathfrak{g}^*)$, which satisfy

$$f_p \equiv f_q \pmod{\alpha_e} \quad (1.4)$$

for every edge, e , of Γ , f_p and f_q being the values of f at the endpoints, p and q , of e . Since f_p and f_q are elements of $\mathbb{S}(\mathfrak{g}^*)$, they can be thought of as polynomial functions on \mathfrak{g} and (1.4) asserts that the restriction of f_p to the hyperplane $\alpha_e = 0$ is equal to the restriction of f_q . More precisely, if $\mathfrak{g}_e^* = \mathfrak{g}^* / \mathbb{R}\alpha_e$, then the projection $\mathfrak{g}^* \rightarrow \mathfrak{g}_e^*$ induces an epimorphism of rings

$$r_e : \mathbb{S}(\mathfrak{g}^*) \rightarrow \mathbb{S}(\mathfrak{g}_e^*)$$

and condition (1.4) can be expressed as

$$r_e(f_{i(e)}) = r_e(f_{t(e)}). \quad (1.5)$$

Since the constant maps of V into $\mathbb{S}(\mathfrak{g}^*)$ satisfy (1.4), $\mathbb{S}(\mathfrak{g}^*)$ is a subring of $H_G(\Gamma)$; so, in particular, $H_G(\Gamma)$ is an $\mathbb{S}(\mathfrak{g}^*)$ -module. Moreover, since it sits inside the free $\mathbb{S}(\mathfrak{g}^*)$ -module (1.3), it is a torsion-free module. It is not, however, *a priori* clear that it is itself a free module; this “freeness” issue is the first of the three questions we will take up below.

The second question concerns the number of generators of $H_G(\Gamma)$. Let $\phi : V \rightarrow \mathbb{R}$ be a Morse function, and, for every vertex, p , let $\text{ind}_p \phi$ be the number of vertices, q , adjacent to p , with $\phi(q) < \phi(p)$. This number, called *the index of p* (and also denoted by σ_p), is the same as the number of oriented edges, e , with $i(e) = p$ and $\alpha_e(\xi) < 0$. We define *the k -th Betti number*, $b_k(\Gamma)$, to be the number of vertices

with $\text{ind}_p \phi = k$ ¹. One can show ([GZ1, Theorem 2.6]) that these numbers don't depend on ξ or ϕ , *i.e.* are graph-theoretic invariants. Our question about the number of generators of $H_G(\Gamma)$ can be formulated as a conjecture:

Conjecture 1. The dimension of the k -th graded component of the ring

$$H_G(\Gamma) \otimes_{\mathbb{S}(\mathfrak{g}^*)} \mathbb{C} \quad (1.6)$$

is equal to $b_k(\Gamma)$.

Remark. It is clear that the orientation o_ξ depends only on the connected component of \mathcal{P} in which ξ sits. On the other hand it is clear that different components will give rise to different orientations. For instance, replacing ξ with $-\xi$ reverses the orientation and changes a vertex of index k into a vertex of index $d - k$. Therefore

$$b_k(\Gamma) = b_{d-k}(\Gamma). \quad (1.7)$$

Finally, what do the generators of the ring $H_G(\Gamma)$ actually look like? For a vertex p of Γ , let F_p be the *flow-up of p* , *i.e.* the set of vertices of the oriented graph (Γ, o_ξ) that can be reached by a path along which the Morse function ϕ is strictly increasing. (F_p is the graph theoretic analogue of the “unstable manifold” at a critical point, p , in ordinary Morse theory.) We also define the *flow-down of p* to be the set of vertices of the oriented graph (Γ, o_ξ) from where we can reach p by a path along which the Morse function ϕ is strictly increasing. Note that the flow-down of p with respect to o_ξ is the flow-up of p with respect to $o_{-\xi}$. Our conjecture is that, just as in ordinary Morse theory, $H_G(\Gamma)$ is generated by cohomology classes dual to these “unstable manifolds”. More explicitly:

Conjecture 2. There exists a set of independent generators

$$\tau^{(p)} \in H_G^k(\Gamma), \quad k = \text{ind}_p \phi \quad (1.8)$$

with $\text{supp } \tau^{(p)} \subset F_p$.

Unfortunately, neither of these conjectures is true without some extra assumptions on (Γ, α) . (An example of a pair (Γ, α) for which both conjectures fail to hold is described in [GZ1, Section 2.1]). Therefore, making a virtue of necessity, we will declare, by definition, that (Γ, α) satisfies *the Morse package* if $H_G(\Gamma)$ is a free $\mathbb{S}(\mathfrak{g}^*)$ -module and if the two conjectures above are true.

We can now formulate the main result of this paper. Let \mathfrak{k} be a linear subspace of \mathfrak{g} , $\mathfrak{h} = \mathfrak{g}/\mathfrak{k}$ and \mathfrak{h}^* its dual, which we can identify with

$$\mathfrak{k}^\perp = \{\gamma \in \mathfrak{g}^* ; \gamma(\eta) = 0 \text{ for all } \eta \in \mathfrak{k}\} \subset \mathfrak{g}^*. \quad (1.9)$$

Define $\Gamma_{\mathfrak{h}^*}$ to be the subgraph of Γ whose edges are all edges, e , of Γ for which $\alpha_e \in \mathfrak{h}^*$.

Theorem. If (Γ, α) satisfies the Morse package, then, for every \mathfrak{h}^* as above, $(\Gamma_{\mathfrak{h}^*}, \alpha)$ satisfies the Morse package. Conversely, if $(\Gamma_{\mathfrak{h}^*}, \alpha)$ satisfies the Morse package for all $\mathfrak{h}^* \subset \mathfrak{g}^*$ of dimension 2, then (Γ, α) satisfies the Morse package.

¹In [GZ1] and [GZ2], we used $b_{2k}(\Gamma)$ to denote this number; also, the k -th homogeneous component of $H_G(\Gamma)$ is denoted there by $H^{2k}(\Gamma, \alpha)$ and here by $H_G^k(\Gamma)$

Example 1. The flag variety, G/B , $G = SL(n, \mathbb{C})$. The graph Γ is a Cayley graph associated with the permutation group, S_n . For this graph, $V = S_n$, and two permutations, σ_k , $k = 1, 2$, are adjacent in Γ if and only if there exists a transposition

$$\tau_{ij} : i \leftrightarrow j, \quad 1 \leq i < j \leq n$$

with $\sigma_2 = \sigma_1 \tau_{ij}$. Moreover, if e is the edge joining σ to $\sigma \tau_{ij}$, the weight labeling e is

$$\alpha_e = \begin{cases} \epsilon_j - \epsilon_i, & \text{if } \sigma(j) > \sigma(i) \\ \epsilon_i - \epsilon_j, & \text{if } \sigma(j) < \sigma(i), \end{cases} \quad (1.10)$$

$\epsilon_1, \dots, \epsilon_n$ being the standard basis vectors of the lattice \mathbb{Z}^n . The function

$$\phi : V \rightarrow \mathbb{Z}, \quad \phi(\sigma) = \text{length}(\sigma) \quad (1.11)$$

is a self-indexing Morse function on V ; our theorem says that for (Γ, α) to satisfy the Morse package, it suffices to check that the subgraphs associated with the subgroups S_2 and S_3 of S_n satisfy the Morse package. This can be done more or less by inspection. For instance the graph associated with S_3 is the permutahedron (see Figure 1) and its “Thom classes” (1.8) are given by simple monomials of degree 1, 2 and 3 in α_1, α_2 and $\alpha_1 + \alpha_2$, with $\alpha_1 = \epsilon_2 - \epsilon_1$ and $\alpha_2 = \epsilon_3 - \epsilon_2$ (see the table below).

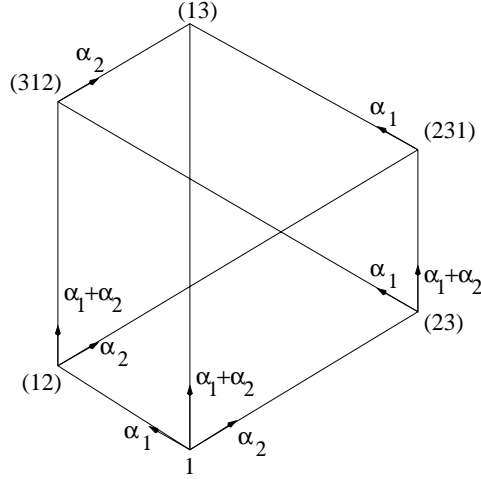


FIGURE 1. Permutahedron

	τ^1	$\tau^{(12)}$	$\tau^{(23)}$	$\tau^{(231)}$	$\tau^{(312)}$	$\tau^{(13)}$
1	1	0	0	0	0	0
(12)	1	$-\alpha_1$	0	0	0	0
(23)	1	0	$-\alpha_2$	0	0	0
(231)	1	$-\alpha_1 - \alpha_2$	$-\alpha_2$	$\alpha_2(\alpha_1 + \alpha_2)$	0	0
(312)	1	$-\alpha_1$	$-\alpha_1 - \alpha_2$	0	$\alpha_1(\alpha_1 + \alpha_2)$	0
(13)	1	$-\alpha_1 - \alpha_2$	$-\alpha_1 - \alpha_2$	$\alpha_2(\alpha_1 + \alpha_2)$	$\alpha_1(\alpha_1 + \alpha_2)$	$-\alpha_1\alpha_2(\alpha_1 + \alpha_2)$

Example 2. The Grassmannian $Gr^k(\mathbb{C}^n)$. Here the graph in question is the *Johnson graph* (see [BCN]). Its vertices are the k -element subsets, S , of the n -element set $\{1, \dots, n\}$, and two vertices, S_k , $k = 1, 2$, are adjacent if $S_1 \cap S_2$ is a $(k-1)$ -element set, or, in other words, if $S_1 - S_2$ and $S_2 - S_1$ are one-element sets. If e is the oriented set joining S_1 to S_2 the weight labeling this edge is

$$\alpha_e = \alpha_i - \alpha_j, \quad (1.12)$$

where $\{i\} = S_1 - S_2$, $\{j\} = S_2 - S_1$ and $\alpha_1, \dots, \alpha_n$ are the standard basis vectors of \mathbb{Z}^n . To define a Morse function on V we first observe that there is a one-to-one correspondence between the vertices of Γ and Young diagrams ([Fu]). Namely, let S be a k -element subset of $\{1, \dots, n\}$, with elements $i_1 < i_2 < \dots < i_k$. Then we make correspond to S the Young diagram, σ_S , with rows of length

$$i_r - r, \quad r = k, k-1, \dots, 1. \quad (1.13)$$

By means of this correspondence we can define a self-indexing Morse function, ϕ , on V , by setting

$$\phi(S) = \text{number of boxes in the diagram } \sigma_S. \quad (1.14)$$

For the Johnson graph, our theorem says that to verify the Morse package for Γ , it suffices to verify the Morse package for the Johnson graph associated with the two-element subsets of $\{1, 2, 3\}$, and just as in the previous example, this can be done more or less by inspection.

A few words about the contents of this paper: We mentioned above that the axioms A1-A3 can be translated into axioms on the axial function, α . In Section 2 we will do this and also give more precise definitions of some of the concepts that we have mentioned in this introduction. In Section 3 we will prove the easy part of our main theorem, the “only if” part. The hard part of the theorem, the “if” part, requires some algebraic results about Vandermonde matrices which we will discuss in the Appendix. Another ingredient in our proof is an *integration* operation on graphs. Given an element, f , of the ring (1.3), we define its *integral* over the graph to be the sum

$$\int_{\Gamma} f = \sum_{p \in V} \frac{f_p}{\prod \alpha_e}, \quad (1.15)$$

the denominator in the p^{th} summand being the product over all oriented edges, e , with $i(e) = p$. The expression on the right appears to belong to the quotient field of $\mathbb{S}(\mathfrak{g}^*)$; however, if $f \in H_G(\Gamma)$, one can show that this sum is in $\mathbb{S}(\mathfrak{g}^*)$. Moreover, $H_G(\Gamma)$ has the following duality property:

Proposition 1.1. If $f \in \text{Maps}(V, \mathbb{S}(\mathfrak{g}^*))$ and

$$\int_{\Gamma} fh \in \mathbb{S}(\mathfrak{g}^*)$$

for all $h \in H_G(\Gamma)$, then $f \in H_G(\Gamma)$.

These results are old results of ours from [GZ1], but to prove the main theorem, we will need to generalize them to *hypergraphs*. (A hypergraph is, like a graph, an object consisting of vertices, hyperedges and an incidence relation which tells one when a vertex is incident to a hyperedge. For hypergraphs, however, more than two vertices can be incident to a hyperedge.)

A third ingredient in our proof is a “cross-section” construction for graphs which we already made extensive use of in [GZ2]. A real number, c , is a *critical value* of the Morse function ϕ if $c = \phi(p)$ for some $p \in V$; otherwise, it is a *regular value*. Let c be a regular value and let V_c be the set of oriented edges, e , of Γ , which intersect the level set $\phi = c$, in the sense that $\phi(p) < c < \phi(q)$, $p = i(e)$ and $q = t(e)$ being the vertices of e . Then V_c is the vertex set of a hypergraph, Γ_c , whose edges are the intersections of the level set $\phi = c$ with the connected components of the graphs $\Gamma_{\mathfrak{h}^*}$, \mathfrak{h}^* being a dimension 2 subspace of \mathfrak{g}^* .

The trickiest and most subtle part of our proof is figuring out how to define the cohomology ring of this hypergraph. One can equip Γ_c with an analogue of the axial function, α , (see Section 5) and mimic the definition (1.4) of $H_G(\Gamma)$; however, this turns out *not* to be the right approach. Trial and error show that the right approach is to mimic the definition of $H_G(\Gamma)$ “by duality” in Proposition 1.1. (See Section 4 for details). This approach, however, has one apparent short-coming: The equivariant cohomology theory of graphs that we described above is a *local* theory in the sense that a prospective cohomology class, $f \in \text{Maps}(V, \mathbb{S}(\mathfrak{g}^*))$, belongs to $H_G(\Gamma)$ if and only if it satisfies condition (1.4) at each edge. If we define hypergraph cohomology by duality it is not at all clear that it will be a local theory in this sense. The localization theorem which we prove in Section 5 shows, however, that it is.

Denoting the cohomology ring of Γ_c by $H(\Gamma_c, \alpha_c)$, we show in Section 7 how the proof of the main theorem can be reduced to the following problem: How does the structure of $H(\Gamma_c, \alpha_c)$ change as one passes through a critical value of ϕ ? If c and c' are regular values of ϕ and there is just one critical point, $p \in V_\Gamma$, with $c < \phi(p) < c'$, how are $H(\Gamma_c, \alpha_c)$ and $H(\Gamma_{c'}, \alpha_{c'})$ related? The main technical result of this paper answers this question by showing that the elements of $H(\Gamma_{c'}, \alpha_{c'})$ are expressible in terms of the elements of $H(\Gamma_c, \alpha_c)$ via an “integral” equation of Vandermonde type.

2. THE MAIN RESULT

Let Γ be a regular d -valent graph, V its set of vertices, and, for each vertex $p \in V$, let E_p be the set of oriented edges with initial vertex p . Define the “tangent bundle” of Γ to be (E_Γ, π) , with projection $\pi : E_\Gamma \rightarrow V_\Gamma$, $\pi(e) = i(e)$. Let \mathfrak{g}^* be an n -dimensional real vector space.

Definition 2.1. An *abstract one-skeleton* is a pair (Γ, α) consisting of a d -valent graph, Γ , and a function

$$\alpha : E_\Gamma \rightarrow \mathfrak{g}^*$$

(called *axial function*), satisfying the axioms:

Axiom A1 : For every $p \in V$, the vectors $\{\alpha_e ; e \in E_p\}$ are pairwise linearly independent.

Axiom A2 : If $e \in E_\Gamma$ then

$$\alpha_{\bar{e}} = -\alpha_e.$$

Axiom A3 : Let $e \in E_\Gamma$ with $p = i(e)$ and $p' = t(e)$. Let $E_p = \{e_i ; i = 1, \dots, d\}$ and $E_{p'} = \{e'_i ; i = 1, \dots, d\}$. Then we can order E_p and $E_{p'}$ such that $e_d = e$, $e'_d = \bar{e}$ and

$$\alpha_{e'_i} = \alpha_{e_i} + c_{i,e} \alpha_e, \quad \text{with } c_{i,e} \in \mathbb{R}.$$

If \mathfrak{h}^* is a linear subspace of \mathfrak{g}^* , we define $\Gamma_{\mathfrak{h}^*}$ to be the subgraph of Γ whose edges are all edges e for which $\alpha_e \in \mathfrak{h}^*$. Then $(\Gamma_{\mathfrak{h}^*}, \alpha)$ is an abstract one-skeleton, whose axial function takes values in \mathfrak{h}^* . Note that if \mathfrak{k} is a subspace of \mathfrak{g} and $\mathfrak{h} = \mathfrak{g}/\mathfrak{k}$ then \mathfrak{h}^* is canonically a subspace of \mathfrak{g}^* and

$$\Gamma_{\mathfrak{h}^*} \simeq \Gamma_{(\mathfrak{g}/\mathfrak{k})^*}.$$

Definition 2.2. A vector $\xi \in \mathfrak{g}$ is called *generic* if for every vertex $p \in V$ and every four edges e_1, e_2, e_3 and e_4 in E_p such that $e_1 \neq e_2$, $e_3 \neq e_4$ and $(e_1, e_2) \neq (e_3, e_4)$, we have

$$\frac{\alpha_{e_1}}{\alpha_{e_1}(\xi)} - \frac{\alpha_{e_2}}{\alpha_{e_2}(\xi)} \neq \frac{\alpha_{e_3}}{\alpha_{e_3}(\xi)} - \frac{\alpha_{e_4}}{\alpha_{e_4}(\xi)}.$$

The set of generic elements, \mathcal{P}_0 , is dense in \mathfrak{g} .

From now on, we assume that (Γ, α) is a ξ -acyclic one-skeleton for some generic and polarizing ξ and that $\phi : V \rightarrow \mathbb{R}$ is ξ -compatible. Without loss of generality, we can assume that ϕ is injective. Let $H(\Gamma, \alpha)$ be the $\mathbb{S}(\mathfrak{g}^*)$ -module defined by (1.4).

Definition 2.3. An element $\tau^{(p)}$ of $H(\Gamma, \alpha)$ is called a *generating class at p* if it satisfies the conditions:

1. $\tau^{(p)}$ is supported on the flow-up at p , F_p
2. $\tau^{(p)}(p) = \prod' \alpha_e$

the product in the second condition being over the edges $e \in E_p$ with $\alpha_e(\xi) < 0$. A set $\{\tau^{(p)}\}_{p \in V}$, containing one generating class for each vertex, is called a *generating family*.

Theorem 2.1. The one skeleton (Γ, α) admits a generating family if and only if for every $m \geq 0$,

$$\dim H^m(\Gamma, \alpha) = \sum_{k=0}^d b_k(\Gamma) \lambda_{m-k, n}, \quad (2.1)$$

where

$$\lambda_j = \lambda_{j, n} = \begin{cases} \dim \mathbb{S}^j(\mathfrak{g}^*) & , \text{ if } j \geq 0 \\ 0 & , \text{ if } j < 0. \end{cases}$$

Proof. The equivalence follows from Theorems 2.4.2 and 2.4.4 of [GZ2]. \square

Remark. A generating family is a basis of $H(\Gamma, \alpha)$ as an $\mathbb{S}(\mathfrak{g}^*)$ -module. So, if a generating family exists, $H(\Gamma, \alpha)$ is a free $\mathbb{S}(\mathfrak{g}^*)$ -module.

The main result of this paper is the following theorem:

Theorem 2.2. The one-skeleton (Γ, α) admits a generating family if and only if for every two-dimensional subspace $\mathfrak{h}^* \subset \mathfrak{g}^*$, every connected component of the induced one-skeleton $(\Gamma_{\mathfrak{h}^*}, \alpha)$ admits a generating family.

One also has the following sharpening of the “if” part of this theorem:

Theorem 2.3. Suppose that for every $q \in F_p - \{p\}$, the index of q is strictly greater than the index of p . Then the class $\tau^{(p)}$ is unique.

In view of Theorem 2.1, to prove Theorem 2.2, it suffices to show that for every m , the dimension of $H^m(\Gamma, \alpha)$ is given by (2.1). To compute this dimension, we first realize (Γ, α) as a cross-section of a product one-skeleton $\Gamma^b = \Gamma \times L$ using a process akin to “symplectic cutting” ([Le]). To each cross-section we attach a module and we show that the module we associate to a cross-section “isomorphic” to (Γ, α) is the same as the cohomology ring of (Γ, α) . When the level is just above the lowest vertex, the cross-section is very simple and we determine the associated module directly. Then we determine how this module changes when we pass from one cross-section to another, going over a vertex. Finally, we compute the dimension of $H^m(\Gamma, \alpha)$ by starting with the dimension of the first cross-section and counting the changes that occur until we have reached the cross-section of Γ^b that is the same as (Γ, α) .

3. THE “ONLY IF” PART OF THEOREM 2.2

Let \mathfrak{h}^* be a two-dimensional subspace of \mathfrak{g}^* , \mathfrak{g} be the dual of \mathfrak{g}^* and \mathfrak{k} be the annihilator of \mathfrak{h}^* in \mathfrak{g} . Choose \mathfrak{h} to be a complementary subspace to \mathfrak{k} in \mathfrak{g} ; then we can identify \mathfrak{h}^* with the vector space dual to \mathfrak{h} .

Now let p be a vertex of $(\Gamma_{\mathfrak{h}^*}, \alpha)$ and let τ be the Thom class, in $H(\Gamma, \alpha)$, associated with the flow-up of p . Thus

$$\tau_p = \prod_{i=1}^r \alpha_{e_i}, \quad (3.1)$$

where r is the index of p and e_1, \dots, e_r are the “downward-pointing” edges at p , *i.e.* $\alpha_{e_i}(\xi) < 0$. Suppose that $\alpha_{e_i} \in \mathfrak{h}^*$ for $i = 1, \dots, s$ and $\alpha_{e_i} \notin \mathfrak{h}^*$ for $i = s+1, \dots, r$. Choose $\xi \in \mathfrak{k}$ such that $\alpha_{e_i}(\xi) \neq 0$ for $i = s+1, \dots, r$. Given $f \in \mathbb{S}(\mathfrak{g}^*)$, let $f^\#$ be the restriction of this polynomial function on \mathfrak{g} to the two-dimensional affine subspace

$$\{\xi\} \times \mathfrak{h} \subset \mathfrak{k} \times \mathfrak{h} = \mathfrak{k} \oplus \mathfrak{h} = \mathfrak{g}.$$

Then we can think of $f^\#$ as an element of $\mathbb{S}(\mathfrak{h}^*)$.

Lemma 3.1. Let $V_{\mathfrak{h}^*}$ be the set of vertices of $\Gamma_{\mathfrak{h}^*}$. For $h \in H(\Gamma, \alpha)$, let $h^\# : V_{\mathfrak{h}^*} \rightarrow \mathbb{S}(\mathfrak{h}^*)$ be defined by

$$(h^\#)(p) = (h(p))^\#.$$

Then $h^\# \in H(\Gamma_{\mathfrak{h}^*}, \alpha)$.

Proof. If p and q are in $V_{\mathfrak{h}^*}$ and are joined by an edge e of $\Gamma_{\mathfrak{h}^*}$, then

$$h(p) - h(q) = f\alpha_e$$

for some $f \in \mathbb{S}(\mathfrak{g}^*)$, so that

$$h^\#(p) - h^\#(q) = f^\#(\alpha_e)^\#.$$

However, since $\alpha_e \in \mathfrak{h}^*$, we have $(\alpha_e)^\# = \alpha_e$. □

Consider, in particular, $\tau^\#$. By (3.1),

$$\tau_p^\# = \left(\prod_{i=s+1}^r \alpha_{e_i}^\# \right) \prod_{i=1}^s \alpha_{e_i}.$$

Moreover, for $s+1 \leq i \leq r$,

$$\alpha_{e_i}^\# = c_i + \beta_{e_i},$$

where $c_i = \alpha_{e_i}(\xi) \neq 0$ and $\beta_{e_i} \in \mathfrak{h}^*$. Thus, if c is the product of the c_i 's,

$$\tau_p^\# = c \prod_{i=1}^s \alpha_{e_i} + g, \quad \text{with } g \in \bigoplus_{j \geq s+1} \mathbb{S}^j(\mathfrak{h}^*).$$

Let $\tau_{\mathfrak{h}^*}^{(p)}$ be the homogeneous component of degree s of the cohomology class $c^{-1}\tau^\#$. Then

$$\tau_{\mathfrak{h}^*}^{(p)}(p) = \prod_{i=1}^s \alpha_{e_i}$$

and, since $\tau^\#$ is supported on the flow-up of p in $\Gamma_{\mathfrak{h}^*}$, so is $\tau_{\mathfrak{h}^*}^{(p)}$.

4. CROSS-SECTIONS

We will say that an axial function, α , is 3-independent if, for every $p \in V$ and every triple of edges, $e_i \in E_p$, $i = 1, 2, 3$, the vectors α_{e_i} are linearly independent. In [GZ2] the “if” part of Theorem 2.2 was proved modulo this 3-independence assumption. We will *not* make this assumption here, however, we will make use of a key ingredient in our earlier proof: Fixing a regular value of ϕ , $c \in \mathbb{R} - \phi(V)$, we will define the c -cross-section, V_c , of Γ , to be the set of oriented edges, e , of Γ , with end points $p = i(e)$ and $q = t(e)$, for which

$$\phi(p) < c < \phi(q). \quad (4.1)$$

In [GZ2] we showed that V_c is the set of vertices of a graph, Γ_c , whose edges are defined as follows: For every two dimensional subspace, \mathfrak{h}^* , of \mathfrak{g}^* , and every connected component, $\Gamma_{\mathfrak{h}^*}^0$, of $\Gamma_{\mathfrak{h}^*}$, let $V_c(\Gamma_{\mathfrak{h}^*}^0)$ be the edges of $\Gamma_{\mathfrak{h}^*}$ satisfying (4.1). If α is 3-independent and if the “only if” part of Theorem 2.2 holds, these subsets of V_c are of cardinality 2 and are by definition the edges of Γ_c .

If we drop the 3-independence assumption we can still define these sets to be the *hyperedges* of Γ_c , but the object we get will no longer be a graph but, instead, a *hypergraph*. Nonetheless many of the results of [GZ2] will continue to hold. We will describe some of these results in this section.

Let \mathfrak{g}_ξ^* be the annihilator of ξ in \mathfrak{g}^* , $\mathbb{S}(\mathfrak{g}_\xi^*)$ be the symmetric algebra of \mathfrak{g}_ξ^* and $Q(\mathfrak{g}_\xi^*)$ be the quotient field of $\mathbb{S}(\mathfrak{g}_\xi^*)$. We define a morphism of graded rings

$$\mathcal{K}_c : H(\Gamma, \alpha) \rightarrow \text{Maps}(V_c, \mathbb{S}(\mathfrak{g}_\xi^*)) \quad (4.2)$$

as follows: For $e \in V_c$ let p and q be the vertices of e . Since $\alpha_e(\xi) \neq 0$, the projection $\mathfrak{g}^* \rightarrow \mathfrak{g}_e^*$ maps \mathfrak{g}_ξ^* bijectively onto \mathfrak{g}_e^* , so one has a composite map

$$\mathfrak{g}^* \rightarrow \mathfrak{g}_e^* \leftrightarrow \mathfrak{g}_\xi^*$$

and hence an induced morphism of graded rings:

$$\mathbb{S}(\mathfrak{g}^*) \rightarrow \mathbb{S}(\mathfrak{g}_e^*) \leftrightarrow \mathbb{S}(\mathfrak{g}_\xi^*).$$

If f is in $H(\Gamma, \alpha)$, the images of f_p and f_q in $\mathbb{S}(\mathfrak{g}_e^*)$ are the same by (1.5) and hence so are their images in $\mathbb{S}(\mathfrak{g}_\xi^*)$. We define $\mathcal{K}_c(f)(e)$ to be this common image and call the map \mathcal{K}_c the *Kirwan map*.

Let $\{x, y_1, \dots, y_{n-1}\}$ be a basis of \mathfrak{g}^* such that $x(\xi) = 1$ and $\{y_1, \dots, y_{n-1}\}$ is a basis of \mathfrak{g}_ξ^* . Let $\alpha = \alpha(\xi)(x - \beta(y)) \in \mathfrak{g}^*$ such that $\alpha(\xi) \neq 0$. Then the map

$$\rho_\alpha : \mathbb{S}(\mathfrak{g}^*) \rightarrow \mathbb{S}(\mathfrak{g}_\xi^*),$$

given via the identification $\mathfrak{g}_\xi^* \simeq \mathfrak{g}^*/\mathbb{R}\alpha$, will send x to $x - \alpha/\alpha(\xi) \in \mathfrak{g}_\xi^*$ and y_j to y_j . Therefore ρ_α will send a polynomial $P(x, y) \in \mathbb{S}(\mathfrak{g}^*)$ to the polynomial $P(x - \alpha/\alpha(\xi), y) = P(\beta(y), y) \in \mathbb{S}(\mathfrak{g}_\xi^*)$.

Hence, if $\alpha_e = m_e(x - \beta_e(y))$, then

$$\mathcal{K}_c(f)(e) = f_p(\beta_e(y), y) \in \mathbb{S}(\mathfrak{g}_\xi^*).$$

Let $e \in V_c$, with endpoints $p = i(e)$ and $q = t(e)$, such that $\phi(p) < c < \phi(q)$. If $e_i, i = 1, \dots, d-1$ are the other edges issuing from p and $e'_i, i = 1, \dots, d-1$ are the other edges issuing from q , we define the Thom class of e in Γ , $\tau_e \in H(\Gamma, \alpha)$, by

$$\tau_e(v) = \begin{cases} \prod \alpha_{e_i} & \text{if } v = p \\ \prod \alpha_{e'_i} & \text{if } v = q \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\alpha_i^\# = \mathcal{K}_c(\alpha_i) = \alpha_i - \frac{\alpha_i(\xi)}{\alpha_e(\xi)} \alpha_e$$

and

$$\delta_e = (m_e \mathcal{K}_c(\tau_e))^{-1} = (m_e \prod_{i=1}^{d-1} \alpha_i^\#)^{-1} \in Q(\mathfrak{g}_\xi^*) \quad (4.3)$$

where $m_e = \alpha_e(\xi) > 0$.

We now define the integral operation

$$\int_{\Gamma_c} : \text{Maps}(V_c, Q(\mathfrak{g}_\xi^*)) \rightarrow Q(\mathfrak{g}_\xi^*)$$

as integration with respect to the discrete measure on Γ_c with density δ_e at e , i.e.

$$\int_{\Gamma_c} f = \sum_{e \in V_c} \delta_e f(e) = \sum_{e \in V_c} \frac{f(e)}{m_e \prod_{i=1}^{d-1} \alpha_i^\#}. \quad (4.4)$$

One of the main results in [GZ1] (Theorem 2.5) is :

Theorem 4.1. If $f \in H(\Gamma, \alpha)$ then

$$\int_{\Gamma_c} \mathcal{K}_c(f) \in \mathbb{S}(\mathfrak{g}_\xi^*).$$

We will use this property to associate an $\mathbb{S}(\mathfrak{g}_\xi^*)$ -module to the cross-section.

Definition 4.1. We denote by $H(\Gamma_c, \alpha_c)$ the set of all maps $f : V_c \rightarrow Q(\mathfrak{g}_\xi^*)$ with the property that

$$\int_{\Gamma_c} f \mathcal{K}_c(h) \in \mathbb{S}(\mathfrak{g}_\xi^*)$$

for all $h \in H(\Gamma, \alpha)$.

Remarks:

1. Let $f \in H(\Gamma_c, \alpha_c)$. Then for every edge $e \in \Gamma_c$,

$$f(e) = m_e \int_{\Gamma_c} f \mathcal{K}_c(\tau_e) \in \mathbb{S}(\mathfrak{g}_\xi^*).$$

Therefore $H(\Gamma_c, \alpha_c) \subset \text{Maps}(V_c, \mathbb{S}(\mathfrak{g}_\xi^*))$.

2. Since the Kirwan map (4.2) is a morphism of rings, from Theorem 4.1 follows that the image of the Kirwan map is included in $H(\Gamma_c, \alpha_c)$.

Example 3. Let p be the vertex of Γ on which ϕ attains its minimum and let $c \in \mathbb{R}$ be a regular value such that p is the only vertex below the c -level. Then the c -cross section consists of the oriented edges with initial vertex at p .

For such an edge, e_i , let

$$\alpha_{e_i} = m_i(x - \beta_i(y)),$$

with $m_i = \alpha_{e_i}(\xi)$ and $\beta_i(y) \in \mathbb{S}^1(\mathfrak{g}_\xi^*)$. Consider $\tau : V_c \rightarrow \mathfrak{g}_\xi^*$, $\tau(e_i) = \beta_i(y)$.

Let $g : V_c \rightarrow \mathbb{S}(\mathfrak{g}_\xi^*)$. For $h \in H(\Gamma, \alpha)$, let $P = h(p) \in \mathbb{S}(\mathfrak{g}^*) \simeq \mathbb{S}(\mathfrak{g}_\xi^*)[x]$. Then

$$\int_{\Gamma_c} g \mathcal{K}_c(h) = \left(\prod_{i=1}^d m_i \right)^{-1} \int_{V_c} g P(\tau) \quad ,$$

where the integral on the right hand side is defined as in (8.12) in the Appendix. Therefore, by Lemma 8.4 in the Appendix, g is an element of $H(\Gamma_c, \alpha_c)$ if and only if there exist $g_0, \dots, g_{d-1} \in \mathbb{S}(\mathfrak{g}_\xi^*)$ such that

$$g = \sum_{k=0}^{d-1} g_k \tau^k. \quad (4.5)$$

We conclude that

$$\dim H^m(\Gamma_c, \alpha_c) = \sum_{k=0}^{d-1} \lambda_{m-k, n}.$$

Moreover, let $g \in H(\Gamma_c, \alpha_c)$ be given by (4.5) and consider

$$h = \sum_{k=0}^{d-1} g_k x^k \in \mathbb{S}(\mathfrak{g}_\xi^*)[x] \simeq \mathbb{S}(\mathfrak{g}^*) \subset H(\Gamma, \alpha).$$

Then $g = \mathcal{K}_c(h)$; this proves that for this cross-section, the Kirwan map \mathcal{K}_c is surjective.

5. THE LOCALIZATION THEOREM FOR $H(\Gamma_c, \alpha_c)$

At the beginning of Section 4 we pointed out that the set V_c can be made into the set of vertices of a hypergraph, Γ_c . In this section we will describe the structure of this hypergraph in more detail and give an alternative description of $H(\Gamma_c, \alpha_c)$, which is more in the spirit of the description (1.4) of $H(\Gamma, \alpha)$.

Recall that an edge of Γ_c is a subset of V_c of the form

$$E = V_c(\Gamma_{\mathfrak{h}^*}^0), \quad (5.1)$$

where \mathfrak{h}^* is a 2-dimensional subspace of \mathfrak{g}^* , spanned by axial vectors α_{e_1} and α_{e_2} , $e_i \in E_p$, and $\Gamma_{\mathfrak{h}^*}^0$ is the connected component of $\Gamma_{\mathfrak{h}^*}$ containing p . If one assigns to the edge (5.1) the multiplicity

$$\mu_E = (\text{valence of } \Gamma_{\mathfrak{h}^*}^0) - 1,$$

then Γ_c becomes a $(d-1)$ -valent hypergraph: each vertex is the point of intersection of $d-1$ edges, counting multiplicities. Let us label each of the edges, E , by a non-zero vector, α_E , in the one-dimensional vector space $\mathfrak{h}^* \cap \mathfrak{g}_\xi^*$. This labeling is not

unique since α_E is only defined up to a scalar; however, one can show that it satisfies axioms similar to the axioms A1-A3 of definition 2.1 (see [GZ2, Section 2.1]). Now fix a vector $\gamma \in \mathfrak{g}_\xi^* - 0$ and let Γ_c^γ be the subgraph of Γ_c consisting of the edges, E , of Γ_c , with $\alpha_E \in \mathbb{R}\gamma$.

Lemma 5.1. The hypergraph Γ_c^γ is totally disconnected: no two distinct edges, E_1 and E_2 , of Γ_c^γ , contain a common vertex.

Proof. Let v be a vertex of Γ_c^γ , corresponding to an edge e of Γ . Let E be an edge of Γ_c^γ incident to v and let \mathfrak{h}^* be the two-dimensional subspace of \mathfrak{g}^* such that $E = V_c(\Gamma_{\mathfrak{h}^*}^0)$. Then \mathfrak{h}^* is generated by γ and α_e , therefore is uniquely determined. \square

As in (5.1), let \mathfrak{h}^* be the 2-dimensional subspace of \mathfrak{g}^* spanned by vectors $\alpha_{e_1}, \alpha_{e_2}, e_1, e_2 \in E_p$ and let $\Gamma_{\mathfrak{h}^*}^0$ be the connected component of $\Gamma_{\mathfrak{h}^*}$ containing p .

Definition 5.1. The *Thom class* of $\Gamma_{\mathfrak{h}^*}^0$,

$$\tau = \tau(\Gamma_{\mathfrak{h}^*}^0) \in H(\Gamma, \alpha),$$

is the class which assigns to each vertex q of $\Gamma_{\mathfrak{h}^*}^0$ the product

$$\prod_{e_i \in E_q - E_q(\Gamma_{\mathfrak{h}^*}^0)} \alpha_{e_i}$$

and assigns zero to every other vertex of Γ .

The existence of such a class enables one to define an injection

$$H(\Gamma_{\mathfrak{h}^*}, \alpha) \rightarrow H(\Gamma, \alpha), \quad f \mapsto f\tau.$$

Moreover, from τ one gets an important identity relating integration over Γ_c and integration over the edges (5.1) of Γ_c :

Lemma 5.2. Given $g \in H(\Gamma_{\mathfrak{h}^*}^0, \alpha)$ and $f : V_c \rightarrow \mathbb{S}(\mathfrak{g}_\xi^*)$, one has

$$\int_E f_E \mathcal{K}_c(g) = \int_{\Gamma_c} f \mathcal{K}_c(\tau g), \quad (5.2)$$

where f_E is the restriction of f to E and the Kirwan map \mathcal{K}_c on the left hand side is the map

$$\mathcal{K}_c : H(\Gamma_{\mathfrak{h}^*}^0, \alpha) \rightarrow H((\Gamma_{\mathfrak{h}^*}^0)_c, \alpha_c).$$

We will use Lemma 5.2 to prove:

Theorem 5.1. A map $f : V_c \rightarrow \mathbb{S}(\mathfrak{g}_\xi^*)$ is in $H(\Gamma_c, \alpha_c)$ if and only if its restriction to every edge E of the hypergraph Γ_c is in $H(E, \alpha_c)$.

Proof. Assume first that $f \in H(\Gamma_c, \alpha_c)$ and let E be an edge of Γ_c , given by (5.1). Denote by f_E the restriction of f to E . For every $g \in H(\Gamma_{\mathfrak{h}^*}^0, \alpha)$ we have

$$\int_E f_E \mathcal{K}_c(g) = \int_{\Gamma_c} f \mathcal{K}_c(\tau g) \in \mathbb{S}(\mathfrak{g}_\xi^*),$$

hence $f_E \in H(E, \alpha_c)$.

Conversely, assume that $f_E \in H(E, \alpha_c)$ for every edge E of Γ_c . Let $g \in H(\Gamma, \alpha)$. We need to show that

$$\int_{\Gamma_c} f \mathcal{K}_c(g) \in \mathbb{S}(\mathfrak{g}_\xi^*). \quad (5.3)$$

where, according to (4.4),

$$\int_{\Gamma_c} f \mathcal{K}_c(g) = \sum_{e \in V_c} \delta(e) f(e) \mathcal{K}_c(g)(e). \quad (5.4)$$

Let $\mathcal{M} \in \mathfrak{g}_\xi^*$ be the set of all linear factors that appear in the denominators of $\delta(e)$ (see (4.3)), for all edges e that intersect the c -level. For $\gamma \in \mathcal{M}$, let \mathcal{M}_γ be the multiplicative system in $\mathbb{S}(\mathfrak{g}_\xi^*)$ generated by elements of \mathcal{M} that are not multiples of γ and let $\mathbb{S}(\mathfrak{g}_\xi^*)_\gamma$ be the corresponding localized ring. To show (5.3) it suffices to show that

$$\int_{\Gamma_c} f \mathcal{K}_c(g) \in \mathbb{S}(\mathfrak{g}_\xi^*)_\gamma \quad (5.5)$$

for every $\gamma \in \mathcal{M}$.

Fix $\gamma \in \mathcal{M}$ and let Γ_c^γ be the corresponding subgraph of Γ_c . Let e be a vertex of Γ_c^γ corresponding to an edge of Γ , and let $E = V_c(\Gamma_{\mathfrak{h}^*}^0)$ be an edge of Γ_c^γ incident to e , where \mathfrak{h}^* is a two-dimensional subspace of \mathfrak{g}^* . Then e is an edge in both Γ and $\Gamma_{\mathfrak{h}^*}^0$; the density associated to e when we integrate over Γ_c is given by (4.3) and the density associated to e when we integrate over E is

$$\delta'_e = (m_e \mathcal{K}_c(\tau'_e))^{-1}, \quad (5.6)$$

where τ'_e is the Thom class of e in $\Gamma_{\mathfrak{h}^*}^0$. If $\tau \in H(\Gamma, \alpha)$ is the Thom class of $\Gamma_{\mathfrak{h}^*}^0$ and

$$\mathbf{n}_E(e) = \mathcal{K}_c(\tau)(e) \in \mathcal{M}_\gamma, \quad \text{for all } e \in E \quad (5.7)$$

then the two densities are related by

$$\delta_e = \frac{\delta'_e}{\mathbf{n}_E(e)}. \quad (5.8)$$

Then (5.4) differs from

$$\sum_E \sum_{e \in E} \delta_e f(e) \mathcal{K}_c(g)(e) = \sum_E \sum_{e \in E} \frac{\delta'_e f(e) \mathcal{K}_c(g)(e)}{\mathbf{n}_E(e)} \quad (5.9)$$

by a term in $\mathbb{S}(\mathfrak{g}_\xi^*)_\gamma$ (the first sums on each side are over all edges of Γ_c^γ).

But

$$\sum_{e \in E} \frac{\delta'_e f(e) \mathcal{K}_c(g)(e)}{\mathbf{n}_E(e)} = \frac{1}{\prod_{e \in E} \mathbf{n}_E(e)} \sum_{e \in E} \left(\delta'_e f(e) \mathcal{K}_c(g)(e) \prod_{e' \neq e} \mathbf{n}_E(e') \right), \quad (5.10)$$

and, according to Lemma 8.2 of the Appendix, there exists a polynomial $P_E \in \mathbb{S}(\mathfrak{g}_\xi^*)[X]$ such that

$$\prod_{e' \neq e} \mathbf{n}_E(e') = P_E(\mathbf{n}_E(e)); \quad (5.11)$$

then, since \mathcal{K}_c is a morphism of rings, (5.9) becomes

$$\sum_E \frac{1}{\prod_{e \in E} \mathbf{n}_E(e)} \int_E f_E \mathcal{K}_c(g P_E(\tau)) \in \mathbb{S}(\mathfrak{g}_\xi^*)_\gamma, \quad (5.12)$$

which concludes the proof. \square

6. CUTTING

Let L be a one-valent graph, with two vertices, labeled 0 and 1, and one edge connecting them. Consider the product graph $\Gamma^b = \Gamma \times L$, with the set of vertices V^b ; we define an axial function

$$\alpha^b : E_{\Gamma^b} \rightarrow (\mathfrak{g} \oplus \mathbb{R})^* \simeq \mathfrak{g}^* \oplus \mathbb{R}^* ,$$

by

$$\begin{aligned} \alpha_{(p,0)(q,0)}^b &= \alpha_{pq}, & \text{if } pq \in E_{\Gamma} \\ \alpha_{(p,1)(q,1)}^b &= \alpha_{pq}, & \text{if } pq \in E_{\Gamma} \\ \alpha_{(p,0)(p,1)}^b &= \mathbf{1} \in \mathbb{R}^* ; \end{aligned}$$

(this would correspond to the product action of $G \times S^1$ on $\Gamma^b = \Gamma \times L$).

Then the pair (Γ^b, α^b) is a one-skeleton, acyclic with respect to the generic vector $(\xi, 1) \in \mathfrak{g} \oplus \mathbb{R}$. Choose $a > \phi_{\max} - \phi_{\min} > 0$ and define $\Phi : V^b \rightarrow \mathbb{R}$ by

$$\Phi(p, t) = \phi(p) + at$$

Then Φ is $(\xi, 1)$ -compatible.

Let $c \in (\phi_{\max}, \phi_{\min} + a)$. The cross section V_c^b consists of oriented edges of type $(p, 0)(p, 1)$ and we can naturally identify it with V , the set of vertices of Γ . We can make V_c^b into a graph, Γ_c^b , by joining the points corresponding to $(p, 0)(p, 1)$ and $(q, 0)(q, 1)$ if and only if p and q are joined by an edge of Γ .

Theorem 6.1. If $c \in (\phi_{\max}, \phi_{\min} + a)$ then

$$H(\Gamma_c^b, \alpha_c^b) \simeq H(\Gamma, \alpha)$$

Proof. There is a natural isomorphism of linear spaces $\mathfrak{g}^* \rightarrow (\mathfrak{g} \oplus \mathbb{R})_{(\xi, 1)}^* \subset \mathfrak{g}^* \oplus \mathbb{R}^*$, given by

$$\sigma \longrightarrow (\sigma, -\sigma(\xi) \cdot \mathbf{1}).$$

This induces an isomorphism of rings

$$\rho : \mathbb{S}((\mathfrak{g} \oplus \mathbb{R})_{(\xi, 1)}^*) \rightarrow \mathbb{S}(\mathfrak{g}^*).$$

and, together with the identification $V_c^b \simeq V$, an isomorphism

$$\rho_* : \text{Maps}(V_c^b, \mathbb{S}((\mathfrak{g} \oplus \mathbb{R})_{(\xi, 1)}^*)) \rightarrow \text{Maps}(V, \mathbb{S}(\mathfrak{g}^*)). \quad (6.1)$$

We will show that (6.1) restricts to a bijection

$$\rho_* : H(\Gamma_c^b, \alpha_c^b) \rightarrow H(\Gamma, \alpha).$$

First, let $g \in H(\Gamma_c^b, \alpha_c^b)$. Let e be an edge of Γ , with endpoints $p = i(e)$ and $q = t(e)$ and let Γ_e be the subgraph of Γ^b with vertices $(p, 0), (p, 1), (q, 0)$ and $(q, 1)$. Consider the Thom class of Γ_e in Γ^b . This, by definition, is the map

$$h_e : V^b \rightarrow \mathbb{S}((\mathfrak{g} \oplus \mathbb{R})^*)$$

that is 0 at vertices not in Γ_e and, at each vertex, v , of Γ_e , is equal to the product of the values of α^b on edges at v which are not edges of Γ_e . Then

$$\frac{\rho_*(g)(p) - \rho_*(g)(q)}{\alpha_e} = \rho \left(\int_{\Gamma_c^b} g \mathcal{K}_c(h_e) \right) \in \mathbb{S}(\mathfrak{g}^*).$$

Since this is true for all edges of Γ , it follows that $\rho_*(g) \in H(\Gamma, \alpha)$.

Conversely, let $g \in H(\Gamma, \alpha)$ and $h \in H(\Gamma^b, \alpha^b)$. The projection on the first factor $\mathfrak{g}^* \oplus \mathbb{R}^* \rightarrow \mathfrak{g}^*$ induces a map $\pi_1 : \mathbb{S}((\mathfrak{g} \oplus \mathbb{R})^*) \rightarrow \mathbb{S}(\mathfrak{g}^*)$. Let

$$h_0 = \pi_1 \circ h|_{\Gamma \times \{0\}};$$

then $h_0 \in H(\Gamma, \alpha)$ and therefore $gh_0 \in H(\Gamma, \alpha)$.

A direct computation shows that

$$\rho\left(\int_{\Gamma_c^b} \rho_*^{-1}(g) \mathcal{K}_c(h)\right) = \int_{\Gamma} gh_0.$$

Theorem 2.2 in [GZ1] implies that the right hand side is an element of $\mathbb{S}(\mathfrak{g}^*)$ and hence

$$\int_{\Gamma_c^b} \rho_*^{-1}(g) \mathcal{K}_c(h) \in \mathbb{S}((\mathfrak{g} \oplus \mathbb{R})_{(\xi, 1)}^*),$$

for all $h \in H(\Gamma, \alpha)$. Therefore

$$\rho_*^{-1}(g) \in H(\Gamma_c^b, \alpha_c^b),$$

which proves the theorem. \square

7. THE CHANGES IN COHOMOLOGY

Let c and c' be regular values of ϕ such that $c < c'$ and such that there is exactly one vertex, p , with $c < \phi(p) < c'$. Let r be the index of p , and let $s = d - r$. Let e_i , $i = 1, \dots, r$ be the edges, e , issuing from p , for which $\alpha_{p,e}(\xi) < 0$ and let e_a , $a = r + 1, \dots, d$ be the other edges issuing from p . We assume that $1 \leq r \leq d - 1$, so that $d - 1 \geq s \geq 1$. Let $\Delta_c = \{e_i ; i = 1, \dots, r\} \subset V_c$ and $\Delta_{c'} = \{e_a ; a = r + 1, \dots, d\} \subset V_{c'}$. Define

$$V^\# = (V_c - \Delta_c) \cup (\Delta_c \times \Delta_{c'})$$

and a map

$$\pi_c : V^\# \rightarrow V_c$$

which is the identity on $V_c - \Delta_c$ and the projection on the first factor on $\Delta_c \times \Delta_{c'}$. Note that

$$V^\# = (V_{c'} - \Delta_{c'}) \cup (\Delta_c \times \Delta_{c'})$$

and that there exists a similar map

$$\pi_{c'} : V^\# \rightarrow V_{c'}.$$

We define the pull-back maps

$$\pi_c^* : \text{Maps}(V_c, \mathbb{S}(\mathfrak{g}_\xi^*)) \rightarrow \text{Maps}(V^\#, \mathbb{S}(\mathfrak{g}_\xi^*))$$

and

$$\pi_{c'}^* : \text{Maps}(V_{c'}, \mathbb{S}(\mathfrak{g}_\xi^*)) \rightarrow \text{Maps}(V^\#, \mathbb{S}(\mathfrak{g}_\xi^*)).$$

The projections of $\Delta_c \times \Delta_{c'}$ onto factors define pull-back maps

$$\text{pr}_1^* : \text{Maps}(\Delta_c, \mathbb{S}(\mathfrak{g}_\xi^*)) \rightarrow \text{Maps}(V^\#, \mathbb{S}(\mathfrak{g}_\xi^*))$$

and

$$\text{pr}_2^* : \text{Maps}(\Delta_{c'}, \mathbb{S}(\mathfrak{g}_\xi^*)) \rightarrow \text{Maps}(V^\#, \mathbb{S}(\mathfrak{g}_\xi^*)),$$

by extending with 0 outside $\Delta_c \times \Delta_{c'}$. Since the maps $\pi_c^*, \pi_{c'}^*, \text{pr}_1^*$ and pr_2^* are injective, we can regard the sets, $\text{Maps}(V_c, \mathbb{S}(\mathfrak{g}_\xi^*))$, $\text{Maps}(V_{c'}, \mathbb{S}(\mathfrak{g}_\xi^*))$, $\text{Maps}(\Delta_c, \mathbb{S}(\mathfrak{g}_\xi^*))$ and $\text{Maps}(\Delta_{c'}, \mathbb{S}(\mathfrak{g}_\xi^*))$ as subsets of $\text{Maps}(V^\#, \mathbb{S}(\mathfrak{g}_\xi^*))$.

Let

$$\begin{aligned} \alpha_{e_i} &= m_i(x - \beta_i(y)) & \text{and} \\ \alpha_{e_a} &= m_a(x - \beta_a(y)), \end{aligned}$$

with $m_i < 0 < m_a$ and $\beta_i(y), \beta_a(y) \in \mathfrak{g}_\xi^*$, for $i = 1, \dots, r$ and $a = r+1, \dots, d$. Consider the maps

$$\begin{aligned} \tau_c : \Delta_c &\rightarrow \mathfrak{g}_\xi^*, & \tau_c(e_i) &= \beta_i(y); \\ \tau_{c'} : \Delta_{c'} &\rightarrow \mathfrak{g}_\xi^*, & \tau_{c'}(e_a) &= \beta_a(y), \text{ and} \\ \tau^\# : \Delta_c \times \Delta_{c'} &\rightarrow \mathfrak{g}_\xi^*, & \tau^\#(e_i, e_a) &= \beta_i(y) - \beta_a(y). \end{aligned}$$

Remark. The fact that ξ is generic implies that $\tau^\#(e_i, e_a) \neq 0$.

Let $H(\Delta_c, \tau_c)$ be the ring associated to the finite set Δ_c and the injective function $\tau_c : \Delta_c \rightarrow \mathfrak{g}_\xi^*$ (see the Appendix, Definition 8.1).

Lemma 7.1. If $f \in H(\Gamma, \alpha)$ then $f|_{\Delta_c} \in H(\Delta_c, \tau_c)$.

Proof. The proof of this “localization” theorem is similar to that of the localization theorem in Section 5. Let $f_0 = f|_{\Delta_c}$ be the restriction of f to Δ_c . To show that $f_0 \in H(\Delta_c, \tau_c)$ we need to show that

$$\int_{\Delta_c} f_0 h(\tau_c) \in \mathbb{S}(\mathfrak{g}_\xi^*) \quad (7.1)$$

for all $h \in \mathbb{S}(\mathfrak{g}_\xi^*)[Y]$, and just as in the proof of Theorem 5.1, we will show that this can be “localized” to analogous statements about the integrals of $f_0 h(\tau_c)$ over the hyperedges of the hypergraph Δ_c . Here are the details.

Let $\mathcal{M} = \{\beta_i - \beta_j \ ; \ i \neq j\} \subset \mathfrak{g}_\xi^*$. For $\gamma \in \mathcal{M}$, let \mathcal{M}_γ be the multiplicative system in $\mathbb{S}(\mathfrak{g}_\xi^*)$ generated by elements of \mathcal{M} which are not collinear with γ and let $\mathbb{S}(\mathfrak{g}_\xi^*)_\gamma$ be the corresponding localized ring. We will show that

$$\int_{\Delta_c} f_0 h(\tau_c) \in \mathbb{S}(\mathfrak{g}_\xi^*)_\gamma \quad (7.2)$$

for each $\gamma \in \mathcal{M}$; this will imply (7.1).

Let $\gamma = \beta_i - \beta_j \in \mathcal{M}$. We join two points v_k and v_l of Δ_c if γ and $\beta_k - \beta_l$ are collinear.

Let $(\Delta_c)_\gamma^0 = \{v_{j_1}, \dots, v_{j_k}\}$ be a non-trivial connected component of this new graph. Then $(\Delta_c)_\gamma^0$ is a complete graph. Let \mathfrak{h}^* be the two-dimensional subspace of \mathfrak{g}^* generated by α_{j_1} and α_{j_2} and let $\Gamma_{\mathfrak{h}^*}^0$ be the connected component of $\Gamma_{\mathfrak{h}^*}$ that contains p . Then

$$(V_{\mathfrak{h}^*}^0)_c \cap \Delta_c = (\Delta_c)_\gamma^0.$$

We now use our hypothesis : $\Gamma_{\mathfrak{h}^*}^0$ admits a generating family. Then the same is true if we change ξ to $-\xi$; let $\tau_{\mathfrak{h}^*}^{(p)} \in H(\Gamma_{\mathfrak{h}^*}, \alpha)$ be a generating class at p with respect to $o_{-\xi}$ and define

$$\psi = \psi_{\mathfrak{h}^*} \tau_{\mathfrak{h}^*}^{(p)} \in H(\Gamma, \alpha),$$

where $\psi_{\mathfrak{h}^*}$ is the Thom class of $\Gamma_{\mathfrak{h}^*}^0$ in Γ . Then ψ is supported on the flow-down of p in $\Gamma_{\mathfrak{h}^*}^0$ and

$$\psi(p) = \prod_{t \notin \{j_1, \dots, j_k\}} \alpha_{e_t}.$$

For every $P \in \mathbb{S}(\mathfrak{g}^*) \simeq \mathbb{S}(\mathfrak{g}_{\xi}^*)[Y]$ we have

$$\int_{(\Delta_c)_{\gamma}^0} f P(\tau_c) = - \left(\prod_{t \in \{j_1, \dots, j_k\}} m_t \right) \int_{\Gamma_c} f \mathcal{K}_c(P\psi); \quad (7.3)$$

since $f \in H(\Gamma_c, \alpha_c)$, (7.3) implies that $f|_{(\Delta_c)_{\gamma}^0} \in H((\Delta_c)_{\gamma}^0, \tau_c)$.

Let $R \in \mathbb{S}(\mathfrak{g}_{\xi}^*)[X]$ be the polynomial

$$R(X) = \prod_{t \notin \{j_1, \dots, j_k\}} (X - \beta_t)$$

and let $T_0 \in \mathbb{S}(\mathfrak{g}_{\xi}^*)[X_1, \dots, X_{k-1}]$ be given by

$$T_0 = \prod_{l=1}^{k-1} R(X_l).$$

Then $T_0 \in (\mathbb{S}(\mathfrak{g}_{\xi}^*)[X_1, \dots, X_{k-1}])^{\Sigma_{k-1}}$, hence, according to Lemma 8.2 of the Appendix, there exists $T \in (\mathbb{S}(\mathfrak{g}_{\xi}^*)[X_1, \dots, X_k])^{\Sigma_k}[Y]$ such that $T_0 = T(X_k)$. In other words, by inserting X_k for Y in this polynomial of $k+1$ variables we get back our original polynomial in $k-1$ variables. Using this we deduce that

$$\sum_{l=1}^k \frac{f(e_{j_l})h(\beta_{j_l})}{\prod_{t \neq j_l} (\beta_{j_l} - \beta_t)} = \frac{1}{\prod_{l=1}^k R(\beta_{j_l})} \int_{(\Delta_c)_{\gamma}^0} (\Psi h)(\tau_c) f, \quad (7.4)$$

where $\Psi \in \mathbb{S}(\mathfrak{g}_{\xi}^*)[Y] \simeq \mathbb{S}(\mathfrak{g}^*)$ is the polynomial obtained by replacing X_l with β_{j_l} , for all $l = 1, \dots, k$.

Since $\prod_{l=1}^k R(\beta_{j_l}) \in \mathcal{M}_{\gamma}$, using (7.3) we conclude that the left hand side of (7.4) is an element of $\mathbb{S}(\mathfrak{g}_{\xi}^*)_{\gamma}$. Now (7.2) follows from the fact that the integral of $f_0 h(\tau_c)$ is a sum of terms of this form. \square

Lemma 7.2. For every $f \in H(\Gamma_c, \tau_c)$ and every $f_i \in H(\Delta_c, \alpha_c)$, $i = 1, \dots, s-1$, there exist unique $f' \in H(\Gamma_{c'}, \alpha_{c'})$ and $f'_j \in H(\Delta_{c'}, \tau_{c'})$, $j = 1, \dots, r-1$, such that

$$f' + \sum_{j=1}^{r-1} (\tau^{\#})^j f'_j = f + \sum_{i=1}^{s-1} (\tau^{\#})^i f_i. \quad (7.5)$$

Proof. Uniqueness: It is clear that f' should be equal to f on $\Gamma_c - \Delta_c = \Gamma_{c'} - \Delta_{c'}$. If we restrict (7.5) to each set of the form $\Delta_c \times \{e_a\}$, we get a linear system whose matrix is a non-singular Vandermonde matrix; therefore, the solution is unique. Hence there exist unique $f' \in \text{Maps}(\Gamma_{c'}, Q(\mathfrak{g}_{\xi}^*))$ and $f'_j \in \text{Maps}(\Delta_{c'}, Q(\mathfrak{g}_{\xi}^*))$ which satisfy (7.5).

Existence: We need to show that $f' \in H(\Gamma_{c'}, \alpha_{c'})$ and $f'_j \in H(\Delta_{c'}, \tau_{c'})$, where f and f'_j are the ones obtained above.

Using Lemma 8.3 and Corollary 8.1 of the Appendix, we deduce that

$$f'_j = \sum_{m'} \left(\sum_{m,k} P_{m,m',k}(y) \left(\int_{\Delta_c} \tau_c^m f_k \right) \right) (\tau_{c'})^{m'},$$

where $P_{m,m',k}(y) \in \mathbb{S}(\mathfrak{g}_\xi^*)$; since $f_k \in H(\Delta_c, \tau_c)$ for all $k = 0, \dots, s-1$, we can use Lemma 8.4 to conclude that $f'_j \in H(\Delta_{c'}, \tau_{c'})$.

Let $h \in H(\Gamma, \alpha)$. Again, we use Lemma 8.3 and Corollary 8.1 to obtain that

$$\int_{\Gamma_{c'}} f' \mathcal{K}_{c'}(h) = \int_{\Gamma_c} f \mathcal{K}_c(h) + \sum_{i=0}^{s-1} \int_{\Delta_c} P_i(\tau_c) f_i,$$

where $P_i \in \mathbb{S}(\mathfrak{g}_\xi^*)[X]$. We now use that $f \in H(\Gamma_c, \alpha_c)$, $f_i \in H(\Delta_c, \tau_c)$ for all $i = 0, \dots, s-1$, and Lemma 8.4 to conclude that

$$\int_{\Gamma_{c'}} f' \mathcal{K}_{c'}(h) \in \mathbb{S}(\mathfrak{g}_\xi^*)$$

for all $h \in H(\Gamma, \alpha)$, i.e. that $f' \in H(\Gamma_{c'}, \alpha_{c'})$. \square

We now use this lemma to determine the change in the cohomology of the cross-section as we pass over the vertex p .

Corollary 7.1. For every $m \geq 0$ we have

$$\dim H^m(\Gamma_{c'}, \alpha_{c'}) = \dim H^m(\Gamma_c, \alpha_c) + \sum_{k=0}^{s-1} \lambda_{m-k, n-1} - \sum_{k=0}^{r-1} \lambda_{m-k, n-1}. \quad (7.6)$$

Proof. When $0 < \sigma(p) < d$, this follows immediately from (7.5) and (8.14). When $\sigma(p) = 0$ it follows from (8.14) and

$$H(\Gamma_{c'}, \alpha_{c'}) = H(\Gamma_c, \alpha_c) \oplus H(\Delta_{c'}, \alpha_{c'})$$

and when $\sigma(p) = d$ it follows from (8.14) and

$$H(\Gamma_c, \alpha_c) = H(\Gamma_{c'}, \alpha_{c'}) \oplus H(\Delta_c, \alpha_c). \quad \square$$

8. ADDING-UP DIMENSIONS

To compute the dimension of $H^m(\Gamma, \alpha)$, we will apply Corollary 7.1 several times to the cross-sections of $\Gamma^b = \Gamma \times L$.

Let $c_0 > \phi_{\min}$ such that there is only one vertex, (p, t) , of $\Gamma^b = \Gamma \times L$, with $\Phi(p, t) < c_0$. Then

$$\dim H^m(\Gamma_{c_0}^b, \alpha_{c_0}^b) = \sum_{k=0}^d \lambda_{m-k, n}.$$

If $c_1 \in (\phi_{\max}, \phi_{\min} + a)$ then, according to Theorem 6.1, we get

$$\dim H^m(\Gamma_{c_1}^b, \alpha_{c_1}^b) = \dim H^m(\Gamma, \alpha).$$

Now let $c_0 < a < b < c_1$ such that there is exactly one vertex of Γ , p , such that $a < \Phi(p, 0) < b$. If the index of p in Γ is $\sigma(p) = r$ then the index of $(p, 0)$ in Γ^b is also r . Thus we can apply (7.6) to obtain

$$\dim H^m(\Gamma_b^b, \alpha_b^b) = \dim H^m(\Gamma_a^b, \alpha_a^b) + \sum_{k=1}^{d-r} \lambda_{m-k, n} - \sum_{k=1}^{r-1} \lambda_{m-k, n}.$$

Adding together these changes we get

$$\dim H^m(\Gamma, \alpha) = \sum_{p \in V} \left(\sum_{k=0}^{d-\sigma(p)} \lambda_{m-k} - \sum_{k=0}^{\sigma(p)-1} \lambda_{m-k} \right)$$

The minimum value for k is 0 and the maximum is d ; λ_{m-k} appears in the first sum when $\sigma(p) \leq d-k$ and in the second one when $\sigma(p) \geq k+1$. Therefore

$$\dim H^m(\Gamma, \alpha) = \sum_{k=0}^d \left(\sum_{l=0}^{d-k} b_l(\Gamma) - \sum_{l=k+1}^d b_l(\Gamma) \right) \lambda_{m-k}.$$

Because $b_{d-l}(\Gamma) = b_l(\Gamma)$ (see (1.7)), the expression in bracket reduces to $b_k(\Gamma)$ and therefore

$$\dim H^m(\Gamma, \alpha) = \sum_{k=0}^d b_k(\Gamma) \lambda_{m-k, n}.$$

This concludes the proof of Theorem 2.2.

Remark. The results of [GZ2, Section 2.5.2] are valid in this case as well. In particular, the dimension of $H^m(\Gamma_c, \alpha_c)$ is the same as the dimension of the image of the Kirwan map

$$\mathcal{K}_c : H^m(\Gamma, \alpha) \rightarrow \text{Maps}(V_c, \mathbb{S}^m(\mathfrak{g}_\xi^*)).$$

Since $\text{im}(\mathcal{K}_c) \subset H(\Gamma_c, \alpha_c)$, it follows that the Kirwan map is surjective. Thus, in particular, $H(\Gamma_c, \alpha_c)$ is not only an $\mathbb{S}(\mathfrak{g}_\xi^*)$ -module, but also a ring.

APPENDIX

Lemma 8.1.

$$\sum_{k=1}^m \frac{X_k^N}{\prod_{j \neq k} (X_k - X_j)} = \sum_{\substack{i_1 + \dots + i_m = N-m+1 \\ i_1, \dots, i_m \geq 0}} X_1^{i_1} \dots X_m^{i_m}. \quad (8.1)$$

Proof. Consider the decomposition in partial fractions

$$\frac{Z^N}{(Z - X_1) \dots (Z - X_m)} = Q(Z) + \sum_{k=1}^m \frac{X_k^N}{\prod_{j \neq k} (X_k - X_j)} \frac{1}{Z - X_k},$$

where $Q(Z)$ is a polynomial in Z . We use the expansion

$$\frac{1}{Z - X_k} = \sum_{i_k=0}^{\infty} X_k^{i_k} Z^{-1-i_k}, \quad (8.2)$$

to obtain that:

$$Z^{N-m} \prod_{k=1}^m \left(\sum_{i_k=0}^{\infty} X_k^{i_k} Z^{-i_k} \right) = Q(Z) + \frac{1}{Z} \sum_{k=1}^m \left(\frac{X_k^N}{\prod_{j \neq k} (X_k - X_j)} \sum_{l=0}^{\infty} X_k^l Z^{-l} \right).$$

Now the formula (8.1) follows by comparing coefficients of Z^{-1} on both sides. \square

Corollary 8.1. Let R be a ring and $P \in R[X_1, \dots, X_m][Y]$. Then

$$\sum_{k=1}^m \frac{P(X_k)}{\prod_{j \neq k} (X_k - X_j)} \in R[X_1, \dots, X_m].$$

Lemma 8.2. Let $P_0 \in (\mathbb{C}[X_1, \dots, X_{m-1}])^{\Sigma_{m-1}}$ be a symmetric polynomial. Then there exists $P \in (\mathbb{C}[X_1, \dots, X_m])^{\Sigma_m}[Y]$ such that $P_0 = P(X_m)$.

Proof. It suffices to show that the lemma is true if P_0 is a fundamental symmetric polynomial in X_1, \dots, X_{m-1} and for this we can use an inductive argument.

We can also start with the identity

$$(Z + X_1) \cdots (Z + X_{m-1}) = \frac{(Z + X_1) \cdots (Z + X_m)}{Z} \frac{1}{1 + X_m Z^{-1}}$$

and we use (8.2) to deduce that

$$(Z + X_1) \cdots (Z + X_{m-1}) = \left(\sum_{k=0}^m \sigma_k Z^{m-k} \right) \left(\sum_{j=0}^{\infty} (-1)^j X_m^j Z^{-j-1} \right),$$

where $\sigma_k \in (\mathbb{C}[X_1, \dots, X_m])^{\Sigma_m}$ is the fundamental symmetric polynomial of degree k . The lemma follows by comparing coefficients of Z^k on both sides. \square

Lemma 8.3. Let $R = \mathbb{Z}[X_1, \dots, X_m]$ and

$$A = \begin{pmatrix} 1 & X_1 & \cdots & X_1^{m-1} \\ 1 & X_2 & \cdots & X_2^{m-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & X_m & \cdots & X_m^{m-1} \end{pmatrix}$$

be a Vandermonde matrix. Then A is invertible in the space of matrices with entries in the quotient field of R and for each $i = 1, \dots, m$, there exists $P_i \in R[Y]$ such that

$$(A^{-1})_{ij} = \frac{P_i(X_j)}{\prod_{k \neq j} (X_j - X_k)}.$$

Moreover, for $i = 1$ we have:

$$(A^{-1})_{1j} = \prod_{k \neq j} \left(\frac{-X_k}{X_j - X_k} \right). \quad (8.3)$$

Proof. Since

$$\det A = \prod_{1 \leq k < l \leq m} (X_l - X_k) \neq 0, \quad (8.4)$$

the matrix A is invertible; the entries of the inverse are

$$(A^{-1})_{ij} = (-1)^{i+j} \frac{\det A_{ji}}{\det A}, \quad (8.5)$$

where A_{ji} is obtained from A by removing the j^{th} row and i^{th} column.

We start with the identity

$$(Z - X_1) \cdots (Z - X_{j-1})(Z - X_{j+1}) \cdots (Z - X_m) = \sum_{k=0}^{m-1} (-1)^{m-1-k} \sigma_{m-1-k}^j Z^k, \quad (8.6)$$

where σ_{m-1-k}^j is the fundamental symmetric polynomial of degree $m-1-k$ in variables $X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_m$. Since the left hand side of (8.6) is 0 for $Z = X_l$, $l \neq j$, we get:

$$X_l^{m-1} = \sum_{k=0}^{m-2} (-1)^{m-k} \sigma_{m-k-1}^j X_l^k. \quad (8.7)$$

The left hand side of (8.7) is the last column of A_{ji} ; using (8.7) and basic properties of determinants we deduce that

$$\det A_{ji} = \sigma_{m-i}^j \prod_{\substack{1 \leq k < l \leq m \\ k \neq j \neq l}} (X_l - X_k). \quad (8.8)$$

Using (8.4) and (8.8) in (8.5) we obtain that

$$(A^{-1})_{ij} = \frac{(-1)^{n+i} \sigma_{n-i}^j}{\prod_{k \neq j} (X_j - X_k)}. \quad (8.9)$$

We now use Lemma 8.2 to finish the proof. Since $\sigma_{n-1}^j = X_1 \cdots X_{j-1} X_{j+1} \cdots X_m$, for $i = 1$, (8.9) becomes (8.3). \square

Remark. The result of Lemma 8.2 can be made more precise. By induction on k we get that

$$\sigma_k^j = \sum_{l=0}^k (-1)^l \sigma_{k-l} X_j^l. \quad (8.10)$$

Then (8.9) becomes:

$$(A^{-1})_{ij} = \frac{\sum_{k=0}^{n-i} (-1)^{n+i+k} \sigma_{n-i-k} X_j^k}{\prod_{k \neq j} (X_j - X_k)}. \quad (8.11)$$

Let W be an n -dimensional vector space, $\Delta = \{v_1, \dots, v_d\}$ be a finite set and $\tau : \Delta \rightarrow W$ be an injective function. Let $\mathbb{S}(W)$ be the symmetric algebra and $Q(W)$ be the quotient field of $\mathbb{S}(W)$.

We define an integral operation $\int_{\Delta} : \text{Maps}(\Delta, Q(W)) \rightarrow Q(W)$ by

$$\int_{\Delta} g = \sum_{k=1}^d \frac{g(v_k)}{\prod_{j \neq k} (\tau(v_k) - \tau(v_j))}. \quad (8.12)$$

Definition 8.1. We define $H(\Delta, \tau)$ to be the space of all maps g satisfying the condition

$$\int_{\Delta} gP(\tau) \in \mathbb{S}(W), \quad \text{for all } P \in \mathbb{S}(W)[Y].$$

Lemma 8.4. A map $g : \Delta \rightarrow Q(W)$ is in $H(\Delta, \tau)$ if and only if there exist $g_0, \dots, g_{d-1} \in \mathbb{S}(W)$ such that

$$g = \sum_{k=0}^{d-1} g_k \tau^k. \quad (8.13)$$

Proof. The fact that every map of the form (8.13) is in $H(\Delta, \tau)$ is a direct consequence of Corollary 8.1.

To show that every element of $H(\Delta, \tau)$ can be written as in (8.13) we proceed as follows: We can regard (8.13) as a linear system whose matrix is a Vandermonde matrix; hence there are $g_0, \dots, g_{d-1} \in Q(W)$ such that (8.13) is true. Moreover, we can use Lemma 8.3 to deduce that

$$g_i = \int_{\Delta} g P_{i+1}(\tau).$$

Since $g \in H(\Delta, \tau)$, we conclude that $g_i \in \mathbb{S}(W)$. □

Remarks:

1. $H(\Delta, \tau)$ is a graded ring and an $\mathbb{S}(W)$ -module.
2. If $g \in H(\Delta, \tau)$ then $g \in \text{Maps}(\Delta, \mathbb{S}(W))$.
3. If $g \in H(\Delta, \tau)$ then the polynomials g_0, \dots, g_{d-1} are unique.
4. Let $H^m(\Delta, \tau) = H(\Delta, \tau) \cap \text{Maps}(\Delta, \mathbb{S}^m(W))$. Then

$$\dim H^m(\Delta, \tau) = \sum_{k=0}^{d-1} \lambda_{m-k, n}. \quad (8.14)$$

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